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LETTER TO THE EDITOR

Q-ball solutions in $1+1$ dimensions for a class of SO(2)-invariant potentials

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Abstract. We obtain exact expressions for the $(1+1)$-dimensional Q ball solutions for a wide class of SO(2)-invariant potentials of the form $\frac{1}{2}\mu^2\phi^2-\frac{1}{6}g^2\phi^{n+2}+\frac{1}{12}\lambda^2\phi^{2n+2}$ where $n=1, 2, 3, \ldots$.

Some time ago Coleman (1985) has shown the existence of the so-called 'Q balls' in a large family of field theories in $3+1$ dimensions when one is near the first-order phase transition point. Actually, his arguments are quite general and Q balls could in fact exist in arbitrary number of dimensions. For example, in a slightly different context, Lee (1976, 1981) has discussed a similar solution in $1+1$ dimensions and has explicitly written down the Q-ball solution for the SO(2)-invariant potential $\frac{1}{2}\mu^2\phi^2-\frac{1}{6}g^2\phi^4+\frac{1}{12}\lambda^2\phi^6$. Recently, Cerveró and Estévez (1986) have also written the Q-ball solution in $1+1$ dimensions for the SO(2)-invariant potential $\frac{1}{2}\mu^2\phi^2-\frac{1}{6}g^2\phi^3+\frac{1}{12}\lambda^2\phi^4$.

The purpose of this letter is to construct exact Q-ball solutions for a class of SO(2)-invariant models characterised by the potential

$$U(\phi)=\tfrac{1}{2}\mu^2\phi^2-\tfrac{1}{6}g^2\phi^{n+2}+\tfrac{1}{12}\lambda^2\phi^{2n+2} \qquad n=1,2,3,\ldots \tag{1}$$

$(\mu^2, g^2, \lambda^2>0)$. It may be noted here that the potentials considered in Lee (1976) and Cerveró and Estévez (1986) are special cases of this general potential for the cases $n=2$ and $n=1$, respectively.

Let us consider the $(d+1)$-dimensional field theory of two real scalar fields ϕ_1 and ϕ_2 defined by the Lagrange density

$$L=\tfrac{1}{2}(\partial_\mu\phi_1)^2+\tfrac{1}{2}(\partial_\mu\phi_2)^2-U(\phi) \tag{2}$$

where $\phi=(\phi_1^2+\phi_2^2)^{1/2}$ and $U(\phi)$ is as given by equation (1). This L has a global SO(2) symmetry; the associated conserved current is

$$j_\mu=\phi_1\partial_\mu\phi_2-\phi_2\partial_\mu\phi_1. \tag{3}$$

Let us further assume that the parameters in $U(\phi)$ have been so chosen (Lee 1976) that there is a Q-ball solution with a finite energy E and finite charge Q of the form

$$\phi_1(x)=\phi(r)\cos\omega t \qquad r=\left(\sum_{i=1}^{d}x_i^2\right)^{1/2} \tag{4a}$$

$$\phi_2(x)=\phi(r)\sin\omega t \tag{4b}$$

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such that $(E - Q\mu) < 0$. Here μ is defined by

$$\mu^2 = (2U/\phi^2)_{\phi=0} = U''(0). \tag{5}$$

The function $\phi(r)$ can then be shown to satisfy the field equation

$$\frac{d^2\phi}{dr^2} + \frac{d-1}{r}\frac{d\phi}{dr} = \frac{dU}{d\phi} - \omega^2\phi. \tag{6}$$

It is not difficult to see that the energy and charge of the Q-ball solution are given by

$$E = \int dr[\tfrac{1}{2}\omega^2\phi^2(r) + \tfrac{1}{2}(\nabla\phi)^2 + U(\phi)] \tag{7}$$

$$Q = \omega \int dr\, \phi^2(r). \tag{8}$$

From here it follows that both $\phi(r)$ and $\nabla\phi$ must vanish sufficiently rapidly as $r \to \infty$ so as to have finite Q and E.

Let us now construct Q-ball solutions in $1+1$ dimensions for the class of potentials given by equation (1). For any even (odd) n, this potential has an absolute minimum at $\phi = 0$ and two (one) local minima at

$$\phi_+^n = \frac{(n+2)}{2(n+1)}\frac{g^2}{\lambda^2}\left[1 + \left(1 - \frac{4(n+1)}{(n+2)^3 c^2}\right)^{1/2}\right] \tag{9}$$

provided

$$\frac{4(n+1)}{(n+2)^2} < c^2 (= g^4/6\lambda^2\mu^2) < 1. \tag{10}$$

On the other hand, at $c^2 = 1$ this potential has three (two) degenerate minima and hence two (one) kink solutions of the form

$$\frac{\phi_1}{\cos\alpha} = \frac{\phi_2}{\sin\alpha} = \phi_k(x) = [(g^2/2\lambda^2)(1 \pm \tanh\tfrac{1}{2}n\mu x)]^{1/n} \tag{11}$$

α being a constant. The corresponding kink mass is

$$E_k = \frac{n\mu}{2(n+2)}\left(\frac{(6\mu)^{1/2}}{\lambda}\right)^{2/n}. \tag{12}$$

On the other hand, when c^2 is as given by equation (10) we have the Q-ball solution

$$\phi_q(x) = \left(\frac{(6\mu c)^{1/2}}{\lambda b}\right)^{1/n}\left(\frac{1}{b + (b^2-1)^{1/2}\cosh(n\mu c/b)x}\right)^{1/n} \tag{13}$$

where

$$b = c(1 - \omega^2/\mu^2)^{-1/2}. \tag{14}$$

It may be noted here that the minimum of $2U(\phi)/\phi^2$ is at $\phi_0^n = g^2/\lambda^2$ and

$$\omega_0^2 \equiv [2U(\phi)/\phi^2]_{\phi_0} = \mu^2(1 - c^2) \tag{15}$$

so that indeed $\omega_0 < \omega < \mu$. From the Q-ball solution (13) it is immediately clear that $\phi_q(x)$ dies off exponentially fast as $x \to \pm\infty$. Further, $\phi_q(x=0)$ is finite while $(d\phi_q(x)/dx)_{x=0}$ vanishes (for any n), so that E and Q are indeed finite for the whole class of Q-ball solutions as given by equation (13).

As expected, the solution (13) for $n=1$ and 2 reduces to the ones given by Cerveró and Estévez (1986) and Lee (1976), respectively. For the $n=2$ case, the Q and E can be shown to be (note $\frac{3}{4}<c^2<1$)

$$Q=\frac{\sqrt{6}\mu}{2\lambda b}(b^2-c^2)^{1/2}\ln\left(\frac{b+1}{b-1}\right) \tag{16}$$

$$E=\frac{\sqrt{6}\mu^2c^2}{4\lambda b^2}(1+b)+\frac{\sqrt{6}\mu^2}{4\lambda}(2-c^2-c^2/b^2)\ln\left(\frac{b+1}{b-1}\right). \tag{17}$$

Let us now discuss the question of energy minimisation as well as charge stability of the solution (13). Since for this solution both E and Q are functions of a single parameter b, E is an implicit function of Q. As a result we find that, except when $n=1$, for our solution E is not minimised for a fixed charge Q. For the special case of $n=1$, it turns out that both E and Q have their minima and maxima for each value of c at the same values of b (Cerveró and Estévez 1986). Our solutions are, however, charge stable since it turns out that for these solutions $E<Q\mu$ for any $b(1<b<\infty)$ provided $c<c_{\text{crit}}$ where c_{crit} is close to one. For example, for $n=1$, $c_{\text{crit}}=0.998$ while for $n=2$, $c_{\text{crit}}=0.984$.

Before concluding this letter we would like to raise the interesting but difficult question of the stability of the Q-ball solutions under small perturbations. We would like to make it clear at the outset that at present we are unable to answer the question in any definite way. Consider the stability of the Q-ball solution (equation (4)) under small perturbations δ_1 and δ_2 defined by (Coleman 1985)

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=\begin{pmatrix}\cos\omega t & -\sin\omega t\\ \sin\omega t & \cos\omega t\end{pmatrix}\begin{pmatrix}\phi_q+\delta_1\\ \delta_2\end{pmatrix} \tag{18}$$

where ϕ_q is the Q-ball solution satisfying the field equation (6). On inserting equation (18) into the equations of motion for ϕ_1 and ϕ_2, retaining only first-order terms in δ_1 and δ_2 and using equation (6) we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta_1-2\omega\frac{\mathrm{d}}{\mathrm{d}t}\delta_2-\nabla^2\delta_1+(U''_q-\omega^2)\delta_1=0 \tag{19}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta_2+2\omega\frac{\mathrm{d}}{\mathrm{d}t}\delta_1-\nabla^2\delta_2+\left(\frac{U'_q}{\phi_q}-\omega^2\right)\delta_2=0 \tag{20}$$

where U_q denotes $U(\phi)$ evaluated at $\phi=\phi_q(x)$. Now the small perturbations δ_1 and δ_2 can be written as

$$\delta_1(x,t)=\exp(\mathrm{i}\omega_1 t)\delta_3(x) \tag{21}$$

$$\delta_2(x,t)=\exp(\mathrm{i}\omega_1 t)\delta_4(x). \tag{22}$$

Then we have the following two coupled equations:

$$[-\nabla^2+(U''_q-\omega^2-\omega_1^2)]\delta_3(x)=2\mathrm{i}\omega\omega_1\delta_4(x) \tag{23}$$

$$[-\nabla^2+(U'_q/\phi_q-\omega^2-\omega_1^2)]\delta_4(x)=-2\mathrm{i}\omega\omega_1\delta_3(x). \tag{24}$$

In order to prove the stability of the Q-ball solution one has to prove that there is no solution to the coupled equations (19) and (20) with $\omega_1=A-\mathrm{i}B$, A and B being real and $B>0$. We have been unable to solve the coupled equations and answer this question in any definite way. It may, however, be worthwhile to point out that there is an exact solution to the coupled equations in the case when $\omega_1 = 0$; the solution has the form

(i) $$d \geqslant 2: \delta_3(\boldsymbol{x}) = \frac{\mathrm{d}\phi_q}{\mathrm{d}r}(r) P_L(\cos\theta) \tag{25a}$$

$$L = \tfrac{1}{2}[-1 + (4d-3)^{1/2}]$$

$$\delta_4(\boldsymbol{x}) = \phi_q(r) \tag{25b}$$

(ii) $$d = 1: \delta_3(x) = \frac{\mathrm{d}}{\mathrm{d}x}\phi_q(x) \tag{26a}$$

$$\delta_4(x) = \phi_q(x). \tag{26b}$$

We notice that $\delta_4(\boldsymbol{x})$ and $\delta_3(\boldsymbol{x})$ are nodeless except when $d = 1$, in which case $\delta_4(x)$ is nodeless while $\delta_3(x)$ is not. As remarked above, this does not help in settling the question of stability of the Q-ball solutions. Finally, notice that one can decouple the two coupled equations (equations (19) and (20)) and obtain uncoupled fourth-order differential equations.

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